PROBLEMS FOR POLYNOMIALS OF ONE AND SEVERAL VARIABLES

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1. Walsh's Coincidence Theorem

Let $z_1, \dots, z_N \in \mathbb{C}$ and $\prod_{k=1}^{k=N}(z+z_k) = \sum_{k=0}^{N} L_k(z_1, \dots, z_N) z^{N-k}$, where L_k stand for standard linear symmetric forms of N variables of degree k. Any linear symmetric form of N variables can be written as

$$L(z_1, \dots, z_N) = \sum_0^N A_K L_k(z_1, \dots, z_N), \ A_k \in \mathbb{C}.$$

When $z_1 = \dots = z_N = z$, we obtain

$$L(z, \dots, z_N) = \sum_0^N \binom{N}{k} A_k z^k, \quad \text{a polynomial in}\, \mathbb{C}.$$

Theorem 1.1. *(**Walsh's coincidence theorem** - cf.* [9]*.) If for $z_1, \dots, z_N) \in \mathbb{D}^N$, the polydisk, a linear symmetric form $L(z_1, \dots, z_N) = 0$, then $\exists z \in \overline{\mathbb{D}} : L(z, \dots, z) = 0$.*

Question. *For which other polynomials P, symmetric with respect to all permutations of the variables, the statement remains true, i.e., if the zero set $\{P = 0\}$ is connected in the polydisk , then it intersects the main diagonal of the polydisk?*

As was noted by S. Shimorin [10], a direct generalization fails for $N \geq 3$ since, e.g., for $P := \sum_1^3 z_j^2 - \sum_{i \neq j} z_i z_j - c$, where $c \neq 0$ is small, the "zero" set of P is a connected "cylinder" in $\mathbb{D}^3$, symmetric with respect to all the permutations of the variables, that does not intersect the main diagonal. It would be interesting to produce a similar example in $\mathbb{D}^2$ of a polynomial $P = P(z_1, z_2)$, such that $P(z_1, z_2) = P(z_2, z_1)$ and the set $\{P = 0\}$ is connected in $\mathbb{D}^2$ but does not intersect the main diagonal. Of course, such an example is impossible in $\mathbb{R}^2$ in view of the intermediate value theorem for continuous functions.

2. Extending the Fundamental Theorem of Algebra

Theorem 2.1. *(**FTA**.) $\forall P(z) := \sum_0^N a_k z^k$, $a_N \neq 0$ has precisely N roots (zeros) in $\mathbb{C}$ counting multiplicities.*

Possible Extensions In late 1980s T. Sheil-Small initiated study of the FTA for harmonic polynomials $h := p(z) - \overline{q(z)}$, such that $n := \deg p > \deg q =: m$. (Such normalization of degrees yields $\sharp\{h = 0\} < \infty$ cf. [9]) Using classical Bezout's theorem, A. Wilmshurst [9] showed that $\sharp\{h = 0\} \leq n^2$ and produced an elegant example showing that n^2 is sharp for all n. m in Wilmshurst's example equals $n - 1$. Let $Z(m, n) := \sharp\{h = 0\}$. D.Khavinson and G. Swiatek [8] showed that $Z(1, n) \leq 3n - 2$, a result conjectured by Sheil-Small and Wilmshurst. L. Geyer [6] proved that this bound is sharp for all n. In 2013, P. Bleher, Y. Homma, L. Ji and R. Roeder [2] showed that the sets $SP_n(k)$ of polynomials p of degree n such that $p - \overline{z}$ has precisely k zeros are all open sets in the space of polynomials of degree n for

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$n \le k \le 3n-2$, $k = n, n+2, \dots, 3n-2$ thus, showing that all these valencies can be assumed with positive probabilities. A. Wilmshurst [9] conjectured that $Z(m,n) \le m(m-1) + 3n - 2$. In the forthcoming paper S. Y. Lee, A. Lerario and E. Lundberg proved that this is not true for all $n \ge 4$ and $m = n-3$.

Question. *Find the asymptotics of $Z(m,n)$ for large m, n. More generally, let $P(x,y) \ge 0$, $deg\, P = n$ has only isolated zeros in $\mathbb{R}^2$. Then the maximal number of real zeros for P is $\le n^2$ and, it is not hard to see using Wilmshurst's example, is $\ge [n^2/4]$. Can any number between these two bounds of real zeros of P occur?*

3. Bohr's radius for polynomials

Theorem 3.1. ***(H. Bohr**, [3]) Let $f(z) = \sum_0^\infty a_k z^k$, $\|f\|_\infty \le 1$ in $\mathbb{D}$. Then, $R_{\mathcal{B}} := \sup\{|z| : \sum |a_k||z|^k \le 1\} = 1/3$ and this bound is sharp.*

(Curiously, Bohr's paper was actually written by G. Hardy based on Bohr's letters, and the sharp bound 1/3 was due in fact, independently, to M. Riesz, I. Schur and F. Wiener.)

Furthermore, ([1],[4]), one has a simple asymptotic for the majorant series

$$M(r) := \sup\{\sum_0^\infty a_k r^k,\ f : \|f\|_\infty \le 1 \,\mathrm{in}\, \mathbb{D}\} \sim \frac{1}{\sqrt{1-r^2}}.$$

Fix $N \ge 1$ and denote by Π_N the set of polynomials $\{P_N(z) := \sum_0^N a_k z^k,\ a_N \ne 0,\ \|P\|_\infty = 1\}$. Define by analogy the n-th Bohr radius $R_N := \sup\{r < 1 : \sum_0^N |a_k| r^k \le 1,\ \forall P \in \Pi_N\}$. Clearly, $R_N > 1/3$.

Question. *What is the rate of convergence of R_N to $1/3$ as $N \to \infty$? In other words, what is the asymptotic of $R_N - \frac{1}{3}$?*

Z. Guadarrama [11] showed that there exist constants C_1, C_2 such that

$$\frac{C_1}{3^{N/2}} \le R_N - \frac{1}{3} \le C_2 \frac{\log N}{N}.$$

R. Fournier [5] showed that R_N equals to the smallest root in $(0,1)$ of the determinant of a certain Toeplitz matrix and, based on some numerical evidence, cautiously suggested that

$$R_N - \frac{1}{3} \sim \frac{\pi^2}{3N^2} + O\left(\frac{1}{N^4}\right).$$

The difficulty of the problem lies with the fact that neither the extremal problem for $R_{\mathcal{B}}$ (D.Khavinson, cf. [11]), nor its counterpart for R_N (R. Fournier, [5]) admit nonconstant solutions.

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